Journal of Novel Applied Sciences
Available online at www.jnasci.org
©2013 JNAS Journal-2013-2-S2/900-905
ISSN 2322-5149 ©2013 JNAS

Product of min-by-max subgroups and relations with residual finite group

B. Razzaghmaneshi

Iranian Fisheries Department of Mathematics and Computer science, Islamic Azad University Talesh Branch, Talesh, Iran Organization, Iran

Corresponding author: B. Razzaghmaneshi

ABSTRACT: *A group G is called min-by-max if it has a normal subgroup N with minimal condition such that the factor group G/N satisfies the maximal condition. And the finite residual J(G) of a group G is the intersection of all normal subgroups of finite index of G. In this paper we show that if $A_1, A_2, \ldots, A_n$ are finitely many pairwise permutable abelian min-by-max subgroups of the group G such that $G=A_1 \ldots A_n$, then G is a soluble min-by-max group and $J(G)=J(A_1)\ldots J(A_n)$.*

Keywords: min-by-max, minimal condition, maximal condition, finite residual group . A ms subject lassification 20.

INTRODUCTION

In 1940 Zappa, (*1940*) and in 1950 Szip, (1950) studied bout products of groups concerned finite groups. In 1961 Kegel, (1961) and in 1958 Wielandt (1958) expressed the famous theorem, whose states the solubility of all finite products of two nilpotent groups .

In 1955 Itô, (1955) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. Cohn, (1956); Zaitsev, (1984) and Redei, (1950); Sesekin, (1968) considered products of cyclic groups, and around 1965 Kegel, (1965) looked at linear and locally finite factorized groups.

In 1968 Sesekin, (1968) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition . He and Amberg independently obtained a similar result for the maximal condition around 1972 (See [17]&[1]). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition $\mathfrak{X}$, when does G have the same finiteness condition $\mathfrak{X}$? (Jetegaonker, 1974)

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of Amberg (Amberg, 1973; Amberg, 1980; Amberg et al., 1991; Amberg et al., 1992; Amberg, 1985); Chernikov, (1980); Kegel, (1961); Lennox, (1980); Robinson, (1986 and 1972); Roseblade, (1965); Sysak, (1982, 1986, 1988 and 1989); Wilson, (1985) and Zaitsev, (1981 and 1984).

Now, in this paper, we study the residual finite group and min-by-max subgroups of the group G and its relations, and the end we prove that if $A_1, A_2, \ldots, A_n$, are finitely many pairwise permutable abelian min-by-max subgroups of the group G such that G is the products of $A_1, \ldots, A_n$. Then G is soluble min-by-max-group and J(G) is products of $J(A_1), \ldots, J(A_n)$, i.e. $J(G)=J(A_1)\ldots J(A_n)$.

***2. Preliminaries**:* (Elementary properties and Theorems.)

In this chapter we express the elementary Lemma and difinitons whose used in prove the Man Theorem in chapter

3. For do this, in chapter 2 we express the elementary lemmas and Theorems and in chapter three we prove the main Theorem.

2.1.Lemma: Let the group G=AB be the product of two subgroups A and B. If x, y are elements of G, then $G=A^xB^y$. Moreover, there exists an element z of G such that $A^x=A^z$ and $B^y=B^z$.

Proof: Write $xy^{-1}=ab$ with a in A and b in B. If $z=a^{-1}x$, then x=az and $y=b^{-1}z,$ so that $A^x=A^z$ and $B^y=B^z$. It follows that $G=A^zB^z=A^xB^y$.

***2.2. Difinition*:** Recall that a finite group is a D_π - groups if evety π - subgroup is contained in a Hall π - subgroup and any two Hall π - subgroups are conjugate.

2.3. Lemma: Let the finite group G=AB be the product of two subgroups A and B. If A,B, and G are D_π - group, for a set π of primes, then there exist Hall π-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Hall π-subgroups of G.

Proof: Let A_1, B_1, and G_1 be Hall π-subgroups of A, B, and G, respectively. Since G is a D_π - group, there exist elements x and y such that A_1^x and B_1^y are both contained in G_1. It follows from Lemma 2.1 that $A^x=A^z$ and $B^y=B^z$ for some z in G. Thus $A_0=A_1^{xz^{-1}}$ and $B_0=B_1^{yz^{-1}}$ are Hall π-subgroups of A and B, respectively, which are both contained in $G_0=G_1^{z^{-1}}$. Clearly the order of $A_0\cap B_0$ is bounded by the maximum π-divisor n of the order of $A\cap B$ since $|G|=\frac{|A|.|B|}{|A\cap B|},$ It follows that

Therefore $A_0B_0=G_0$ is a Hall π-subgroup of G.

2.4. Corollary: Let the finite group G=AB=AK=BK be the product of three nilpotent subgroups, A,B, and K, where K is normal in G. Then G is nilpotent.

Proof: Amberg, 1992, corollary 1.3.5)

2.5. Theorem Itô, 1955: Let the group G=AB be the product of two abelian subgroups A and B. Then G is metabelian.

***Proof*:** Let a,a_1 be elements of A and b, b_1 elements of B. Write $b^{a_1}=a_2b_2$ and $a^{b_1}=b_3a_3$, whit a_2, a_3 in A and b_2,b_3 in B. Then

$$[a,b]^{b_1a_1}=[a,b^{a_1}]^{b_1}=[a,b_2]^{b_1}=[a^{b_1},b_2]=[a_3,b_2]$$

and

$$[a,b]^{b_1a_1}=[a^{b_1},b]^{a_1}=[a_3,b]^{a_1}=[a_3,b^{a_1}]=[a_3,b_2].$$

This proves that the commutators [a,b] and [a_1,b_1] commute. Since the factor group G/[A,B] is abelian, it follows that $G'=[a,b],$ and hence G' is abelian.

2.6. Difinition*:* Recall that the FC-centre of a group G is the subgroup of all elements of G with a finite number of conjugates. A group is an FC-group if it coincides with its FC-centre.

2.7. Lemma: Let the group G=AB be the product of two abelian subgroups A and B, and let S be a factorized subgroup of G. Then the centralizer $C_G(S)$ is factorized . Moreover, every term of the upper central series of G is factorized.

Proof: Since S is factorized, we have that $S=(A\cap S)(B\cap S).$ Let x=ab be an element of S, where a is in $A\cap S$ and b is in $B\cap S$ .If $c=a_1b_1$ is an element of $C_G(S)$, with a_1 in A and b_1 in B, it follows that.

$$[a_1,x]=[a_1,ab]=[a_1,b]=\left[cb_1^{-1},b\right]=[c,b]^{b_1^{-1}}=1.$$

Therefore a_1 belongs to $C_G(S)$, and $C_G(S)$ is factorized by Lemma 1.1.1 of (Amberg et al., 1992). In particular, the center of G is factorized. It follows from Lemma 1.1.2 of [4] that also every term of the upper central series of G is factorized.

2.8. Lemma: Let the group G=AB be the product of two subgroups A and B. If A_1, B_1, and F are the FC-centers of A, B, and C, respectively, then $F=A_1F\cap B_1F$. In particular, if A and B are FC-groups, the FC-centre of G is factorized subgroup.

Proof: Let x be an element of $A_1F\cap B_1F$, and write x=au where a is in A_1 and u is in F. Since the centralizers $C_A(a)$ and $C_A(u)$ have finite index in A, the index $|A: C_A(x)|$ is also finite. Similarly, $C_B(x)$ has finite index in B. Therefore $|G:\langle C_A(x),C_B(x)\rangle|$ is finite by Lemma 1.2.5 of (Amberg et al., 1992). It follows that $C_G(x)$ has finite index in G and hence x belongs to F. Thus $F=A_1F\cap B_1F$.

2.9. Lemma Itô, 1955**:** Let the finite non-trivial group G=AB be the product of two abelian subgroups A and B. Then there exists a non-trivial normal subgroup of G contained in A or B.

Proof: Assume that {1} is the only normal subgroup of G contained in A or B. By Lemma 2.7 have $Z(G)=(A\cap Z(G))(B\cap Z(G))=1.$ The centralizer $C=C_G(A\cap C_G(G'))\, contains AG',$ and so is normal in G. Since $B\cap(AZ(C))\le Z(G)=1,$ it follows that $AZ(C)=A(B\cap AZ(C))=A.$ This Z(G) is a normal subgroup of G contained in A, and so Z(G)=1. Since G' is abelian by Theorem 2.5, we have $A\cap G'\le A\cap C_G(G')\le Z(C)=1.$ Similarly $B\cap G'\le B\cap C_G(G')\le Z(C)=1.$ The factorizer $X=X(G')$ has the triple factorization $X=A^*B^*=A^*G'=B^*G'$, Where $A^*=A\cap BG'$ *and* $B^*=B\cap AG'.$ Thus X is nilpotent by Corollary 2.4, so that

$$Z(X)=(A\cap Z(X))(B\cap Z(X))$$

is not trivial. Hence there exists a non-trivial normal subgroup N of X contained in A or B. Suppose that N is contained in A. Since G' normalizes N, we have $[N,G']\le N\cap G'\le A\cap G'=1.$ Therefore we obtain the contradiction $N\le A\cap G_G(G')=1.$

2.10. Corrollary: Let the finite group $G=A_1\ldots A_t$ be the product of pairwise permutable nilpotent subgroups $A_1,\ldots,A_t$. Then G is soluble.

Proof: Let p be a prime, and for every i=1…,t let P_1 be the unique Sylow p-complement of A_i. If $i\neq j$, the subgroup A_iA_j is soluble by Theorem 2.4.3 of [4]. Hence it follows from Lemma 2.3,that P_iP_j is a Sylow p-complement of A_iA_j. Thuse the subgroups $P_1,\ldots,P_t$ pairwise permute, and the product $P_1P_2\ldots P_t$ is a Sylow p-complement of G. Since G has a Sylow p-complement for every prime p, it is soluble.

2.11. Theorem (Zaitsev, 1981; Lennox and Roseblade, 1980)**:** If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

Proof: Assume that G it not polycyclic-by-finite. Then G contains an abelian normal section U/V which is either torsion-free or periodic and is not finitely generated. Clearly the factorizer of U/V in G/V is also a counterexample. Hence we may suppose that G has a triple factorization G=AB=AK=BK, Where K is an abelian normal subgroup of G which is either torsion-free or periodic. By Lemma 1.2.6(i) of (Amberg et al., 1992; Jetegaonker, 1974) the group G satisfies the maximal condition on normal subgroups, so that it contains a normal subgroup M which is maximal with respect to the condition that G/M is not polycyclic-by-finite. Thus it can be assumed that every proper factor group of G is polycylic-by-finite.

2.12. Theorem (Robinson, 1972): Let the soluble group G=AB be the product of two subgroups A and B with finite abelian section rank. If at least one of the factors A and B has an ascending normal series with central or periodic factors, then G also has finite abelian section rank.
Proof: Theorem 4.6.10) .

2.13. Theorem: Let the group G=AB=AK=BK be the product of three nilpotent subgroups A, B, and K, where K is normal in G. If K is minimax, then G is nilpotent.

Proof : Theorem 6.3.4.

2.14.Theorem (See [6])**:** Let the group G=AB=AK=BK be the product of two subgroups A and B and a minimax normal subgroup K of G.
(i) if A,B, and K are locally nilpotent, then G is locally nilpotent.
(ii) If A, B, and K are hypercentral, then G is hypercentral.

Proof: Theorem 6.3.7

2.15. Lemma: Let the group G=AB be the product of two abelian subgroups A and B such that $A_G=B_G=1$. Then the following hold .
(i) $A\cap B=Z(G)=1.$
(ii) $A\cap C_G(G')=B\cap C_G(G')=1,$ and in particular $A\cap G'=B\cap G'=1.$
(iii) The factorizer $X=X(G')$ of G' does not have non-trivial normal subgroups which are contained in A or B, so that in particular Z(X)=1.
(iv) The FC-centre of G is trivial.

Proof : (i) They Lemma 2.7 we have that

$$Z(G)=(A\cap Z(G))(B\cap Z(G)); A_G B_G=1.$$

Hence Z(G)=1. Moreover, $A\cap B$ in contained in Z(G) and so is also trivial.
(ii)This follows from the first part of the proof of Lemma 2.9.
(iii) Let N be a normal subgroup of X contained in A.Then G' normalizes N, so that by (ii)

$$[N,G']=N\cap G=A\cap G'=1$$

Therefore N is contained in $A\cap C_G(G')=1$

(iv) Let a be an element of $A\cap F$, where F is the FC-centre of G. Since G' is abelian by Theorem 2.5, the mapping $\varphi: x\mapsto[x,a]$ is a G epimorphism from G' onto $[G',a]$. Hence $C_{G'}(a)=\ker\varphi$ is a normal subgroup of G, and the abelian groups $G'/C_{G'}(a)$ and $[G',a]$ are G isomorphic. The factorizer X=X(G') of G' has the triple factorization
X=A*B*=A*G'=B*G',
Where $A^*=A\cap BG'$ and $B^*=B\cap AG'$. As $G'/C_{G'}(a)$ is finite, it follows from Theorem 2.13 that $X/C_{G'}(a)$ is nilpotent. Therefore $[G',a]$ is contained in some term of the upper central series of X. Since Z(X)=1 by(iii), we have $[G',a]$=1 and so a belongs to $A\cap C_G(G').$ Thus a=1 by (ii), and hence $A\cap F=1.$ Similarly $B\cap F=1.$ It follows from Lemma 2.8 that $F=(A\cap F)(B\cap F)=1.$

2.16. Theorem: Let the group G=AB$\neq 1$ be the product of two abelian subgroups A and B, at least one of which has finite section rank. Then there exists a non-trivial normal subgroup of G contained in A or B.
Proof: Assume that $A_G=B_G=1,$ so that $A\cap G'=B\cap G'=1.$ by Lemma 2.15(ii). The factorizer $X=X(G')$ has the triple factorization

$$X=(A\cap BG')(B\cap AG')=(A\cap BG')G'=(B\cap AG')G',$$

And its centre is trivial by Lemma 2.15(iii). The subgroups $A\cap BG'$ and $B\cap AG'$ are isomorphic, and hence both have finite section rank. By Theorem 2.12 the metabelian group X has finite abelian section rank, and hence is hypercentral by Theorem 2.14. In particular $Z(X)\neq 1,$ a contradiction.

2.17. Theorem: (See [35])**:** Let the group $G=A_1\ldots A_t$ be the product of finitely many pairwise permutable abelian minimax subgroups $A_1,\ldots,A_t$. Then G is a soluble minimax group.

Proof: Assume that the theorem is false, and let $G= A_1\ldots A_t$ be a counterexample for which the sum $t+\sum_{i=1}^{t} m(A_i)$ is minimal. Suppose that there are indices i<j such that $D=A_i\cap A_j$ is infinite. Then

$$D^G = D^{A_1\ldots A_t} = D^{A_1\ldots A_{i-1}A_{i+1}\ldots A_{j-1}A_{j+1}\ldots A_t} \leq A_1\ldots A_i\ldots A_{j-1}A_{j+1}\ldots A_t.$$

It follows that D^G is a soluble minimax group. On the other hand, the factor group $\overline{G}=G/D^G$ is also a soluble minimax group since $m(\overline{A}_i)<m(A_i)$. This contradiction shows that $A_i\cap A_i$ is finite if $i\neq j$.

Let J_i be the finite residual of A_i for every i=1,…,t. It follows from lemma 2.15 that J_iJ_j is the finite residual of the soluble minimax group A_iA_j, so that it is abelian . Hence $L=\langle J_1,\ldots,J_t\rangle$ is an abelian group satisfying the minimal condition. As $[A_i,J_j]\leq J_iJ_j\leq L,$ the subgroup L is normal in G. Assume that $J_i\neq 1$ for some i. Then $m(A_iL/L)<m(A_i),$ and so G/L is a soluble minimax group. This contradiction proves that $J_i=1$ for each i. In particular the maximum periodic normal subgroup E of A_1A_2 is finite. If A_1A_2=E, then the soluble minimax group by Corollary 2.10 Thus E is properly contained in A_1A_2, and by Theorem 2.16 we may suppose that A_1E/E contains a non-trivial normal subgroup N/E of
$A_1A_2/E=(A_1E/E)(A_2E/E)$.
As A_1A_2/E has no finite-non-trivial normal subgroups, N/E must be infinite . Moreover, the index $|N:N\cap A_1|=|A_1N:A_1|\leq|A_1E:A_1|$ is finite. If M is the core of $N\cap A_1$ in A_1A_2, then N/M has finite exponent and hence is finite. Therefore M is an infinite normal subgroup of A_1A_2 contained is A_1. Since $M^G=M^{A_3\ldots A_t}\leq A_1A_3\ldots A_t$, it follows that M^G is a soluble minimax group. As above, G/M^G is also a soluble minimax group since $m(A_1M^G/M^G)<m(A1)$ This contradiction proves the theorem.

Chapter 3: Proof of the main Theorem: In this chapter by used the Lemmas and Theorems of chaper 2, we prove the Basic theorem of this paper as follows.

3.1. Main Theorem: Let the group $G= A_1\ldots A_t$ be the product of finitely many pairwise permutable abelian min-by-max subgroups $A_1,\ldots,A_t$. Then G is a soluble min-by-max group and $J(G)=J(A_1)\ldots J(A_t)$.

Proof: It follows from Theorem 2.17 that G is soluble minimax group, and hence J=J(G) is abelian. Put $J_i=J(A_i)$ for each i=1,…,t. Then $L=J_1\ldots J_t$ is contained in J. Let I be the finite residual of A_iA_j. The factorizer X=X(I) of I in A_iA_j has the triple factorization $X=A_i^*A_j^*=A_i^*I=A_j^*I,$
where $A_i^*=A_i\cap A_jI$ and $A_j^*=A_j\cap A_iI$. It follows that J_i and J_j are contained in Z(X), and the factor group X/ J_iJ_j is polycyclic by Theorem2.11. Therefore J_iJ_j is the finite residual of X and so $J_iJ_j=I.$ Thus $[A_i,J_j]\leq J_iJ_j\leq L,$ and hence L is normal in G. The factor group A_iL/L is polucyclic for every $i\leq t$, and hence also $G/L=(A_1L/L)\ldots(A_tL/L)$ is polycyclic by Theorem 2.10. This proves that G is a min-by-max group and $J=L=J_1\ldots J_t$.

REFERENCES

Amberg B. 1973. Factorizations of Infinite Groups. Habilitationsschrift, Universität Mainz.
Amberg B. 1980. Lokal endlich-auflösbare Produkte von zwei hyperzentralen Gruppen. Arch. Math. (Basel) 35, 228-238.
Ambrg B, Franciosi S and de Giovanni F. 1991. Rank formulae for factorized groups. Ukrain. Mat. Z. 43, 1078-1084.
Amberg B, Franciosi S and de Gioranni F. 1992. Products of Groups. Oxford University Press Inc., New York.
Chernikov NS. 1980. Factorizations of locally finite groups. Sibir. Mat. Z. 21, 186-195. (Siber. Math. J. 21, 890-897.)
Amberg B. 1985. On groups which are the product of two abelian subgroups. Glasgow Math. J. 26, 151-156.
Itô N. 1955. Über das Produkt von zwei abelschen Gruppen. Math.Z. 62, 400-401.
Kegel OH. 1961. Produkte nilpotenter Gruppen. Arch. Math. (Basel) 12, 90-93.
Robinson DJS. 1986. Soluble products of nilpotent groups. J. Algebra 98, 183-196.
Wielandt H. 1958. Über Produkte von nilpotenten Gruppen. Illinois J. Math. 2, 611-618.
Zaitsev DI. 1981. Factorizations of polycyclic groups. Mat. Zametki 29, 481-490. (Math. Notes 29, 247-252).
Lennox JC and Roseblade JE. 1980. Soluble products of polycyclic groups. Math. Z. 170, 153-154.
Roseblade JE. 1965. On groups in which every subgroup is subnormal. J. Algebra 2, 402-412.
Kovacs LG. 1968. On finite soluble groups. Math. Z. 103, 37-39.
Robinson DJS. 1972. Finiteness Conditions and Generalized Soluble Groups. Springer, Berlin.
Kegel OH and Wehrfritz BAF. 1973. Locally Finite Groups. North-Holland, Amsterdam.
Jetegaonker AV. 1974. Integral group rings of polycyclic-by-finite groups. J. Pure Appl. Algebra 4, 337-343.
Zaitsev DI. 1984. Soluble factorized groups. In Structure of Groups and Subgroup Characterizations, pp. 15-33. Akad. Nauk Ukrain. Inst. Mat. Kiev (in Russian).
Sesekin NF. 1968. Product of finitely connected abelian groups. Sib. Mat. Z. 9, 1427-1430. (Sib. Math. J. 9, 1070-1072.)
Sesekin NF. 1973. On the product of two finitely generated abelian groups. Mat. Zametki 13, 443-446. (Math. Notes 13, 266-268)
Cohn PM. 1956. A remark on the general product of two infinite cyclic groups. Arch. Math. (Basel) 7, 94-99.
Redei L. 1950. Zur Theorie der faktorisierbaren Gruppen I. Acta Math. Hungar. 1, 74-98.
Szep J. 1950. On factorisable, not simple groups. Acta Univ. Szeged Sect. Sci. Math. 13, 239-241.
Zappa G. 1940. Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro. In Atti del Secondo Congresso dell'Unione Matematica Italiana, pp. 119-125. Cremonese, Rome.
Zatisev DI. 1980. Products of abelian groups. Algebra i Logika 19, 150-172. (Algebra and Logic 19, 94-106.)
Kegel OH. 1965. Zur Struktur mehrfach faktorisierter endlicher Grouppen. Math. Z. 87, 42-48.
Kegel OH. 1965. on the solvability of some factorized linear groups. Illinois J.Math. 9, 535-547.
Franciosi S and de Giovanni F. 1990. On products of locally polycyclic groups. Arch. Math. (Basel) 55, 417-421.
Franciosi S and de Giovanni F. 1990. On normal subgroups of factorized groups. Ricerche Mat. 39, 159-167.
Franciosi S and de Giovanni F. 1992. On trifactorized soluble of finite rank. Geom. Dedicata 38, 331-341.
Franciosi S and de Giovanni F. 1992. On the Hirsch-Plotkin radical of a factorized group. Glasgow Math. J. To appear.
Franciosi S, de Giovanni F, Heineken H and Newell ML. 1991. On the Fitting length of a soluble product of nilpotent groups. Arch. Math. (Basel) 57, 313-318.
Sysak YP. 1982. Products of infinite groups. Preprint 82.53, Akad. Nauk Ukrain. Inst. Mat. Kiev (in Russian).
Sysak YP. 1986. Products of locally cyclic torsion-free groups. Algebra i Logika 25, 672-686. (Algebra and Logic 25, 425-433.)
Sysak YP. 1988. On products of almost abelian groups. In Researches on Groups with Restrictions on Subgroups, pp. 81-85. Akad . Nauk Ukrain. Inst. Mat. Kiev (in Russian).
Sysak YP. 1989. Radical modules over groups of finite rank. Preprint 89.18, Akad. Nauk Ukrain. Inst. Mat., Kiev (in Russian).
Wilson JS. 1985. On products of soluble groups of finite rank. Comment . Math. Helv. 60, 337-353.
Tomkinson MJ. 1986. Products of abelian subgroups. Arch. Math. (Basel) 42, 107-112.